Supplementary Midsemestral Examination
Algebra III - B. Sury
November 26, 2019
Answer ANY SIX.

Q 1. Let R be any ring in which the equation $ax = b$ has solutions for any $a \neq 0$ and $b \in R$. Prove:
(i) R has no (left or right) zero-divisors other than 0,
(ii) R has a unity,
(iii) R is a division ring.

Q 2. Call a complex number z an algebraic integer, if it is a root of a monic polynomial $p(x)$ with integer coefficients. Consider the subring $\overline{\mathbb{Z}}$ of $\mathbb{C}$ consisting of all algebraic integers. Show that there exists a strictly increasing sequence of ideals in $\overline{\mathbb{Z}}$.
Hint. Think of real 2^n-th roots of 2.

Q 3. For any positive integer n, give an explicit isomorphism between the ring $\mathbb{Z}[i]/(1+in)$ and $\mathbb{Z}/(1+n^2)$.

Q 4. Let X be an infinite set. Consider

$$I := \{A \subset X : |A| < \infty\}.$$

Prove that I is an ideal of $P(X)$ and that it is not of the form $P(Y)$ for any subset Y of X.

Q 5. We have already seen that a Boolean ring (ring in which each element satisfies $x^2 = x$) must be commutative of characteristic 2. Show that every ideal generated by finitely many elements $a_1, \cdots, a_n$ in a Boolean ring, must be generated by a single element.

Q 6. In the ring $C([0,1], \mathbb{R})$ of continuous, real-valued functions on $[0,1]$, show that an ideal generated by elements $f_1, \cdots, f_n$ is the full ring if and only if the functions $f_1, \cdots, f_n$ have no common zero in $[0,1]$.

Q 7.

(a) Give an example of a commutative ring R in which for each positive integer k, there exists an element a such that $a^k = 0$ but there is no fixed n such that $r^n = 0$ for all $r \in R$.

Hint: Think of a ring of the form $R_1 \times R_2 \times \cdots$

(b) Let R be any commutative ring which has only finitely many nilpotent elements. Prove that any power series $\sum_{n=0}^{\infty} a_n X^n$ in $R[[X]]$ all of whose coefficients are nilpotent in R, is a nilpotent element of $R[[X]]$.